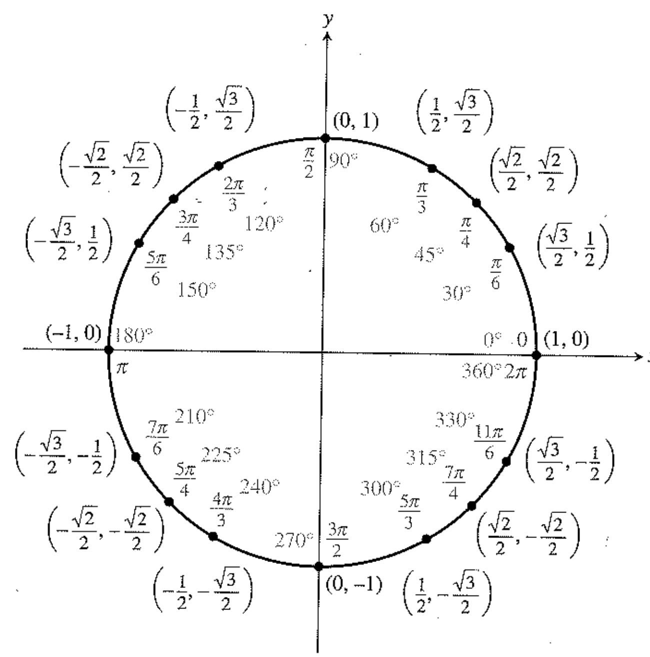
We take a closer look at the properties of the six circular functions in the next two sections.

The 16-Point Unit Circle

At this point you should be able to use reference triangles and quadrantal angles to evaluate trigonometric functions for all integer multiples of 30° or 45° (equivalently, $\pi/6$ radians or $\pi/4$ radians). All of these special values wrap to the 16 special points shown on the unit circle below. Study this diagram until you are confident that you can find the coordinates of these points easily, but avoid using it as a reference when doing problems.



QUICK REVIEW 4.3 *(For help, go to Section 4.1.)*

In Exercises 1–4, give the value of the angle θ in degrees.

1. $\theta = -\frac{\pi}{6}$ **2.** $\theta = -\frac{5\pi}{6}$

3. $\theta = \frac{25\pi}{4}$ **4.** $\theta = \frac{16\pi}{3}$

In Exercises 5–8, use special triangles to evaluate:

5. $\tan \frac{\pi}{6}$ **6.** $\cot \frac{\pi}{4}$

7. $\csc \frac{\pi}{4}$ **8.** $\sec \frac{\pi}{3}$

In Exercises 9 and 10, use a right triangle to find the other five trigonometric functions of the *acute* angle θ.

9. $\sin \theta = \frac{5}{13}$ **10.** $\cos \theta = \frac{15}{17}$

SECTION 4.3 EXERCISES

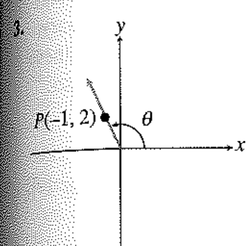
In Exercises 1 and 2, identify the one angle that is not coterminal with all the others.

1. 150°, 510°, −210°, 450°, 870°

2. $\frac{5\pi}{3}, -\frac{5\pi}{3}, \frac{11\pi}{3}, -\frac{7\pi}{3}, \frac{365\pi}{3}$

In Exercises 3–6, evaluate the six trigonometric functions of the angle θ.

3.



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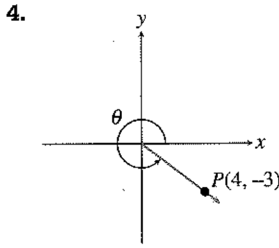

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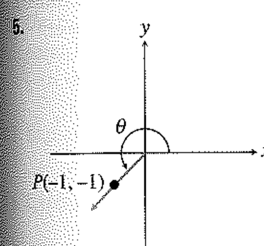

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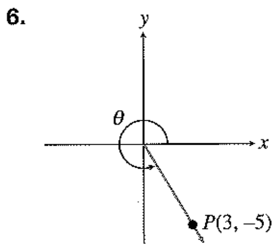


In Exercises 7–12, point P is on the terminal side of angle θ. Evaluate the six trigonometric functions for θ. If the function is undefined, write "undefined."

7. $P(3, 4)$ **8.** $P(-4, -6)$

9. $P(0, 5)$ **10.** $P(-3, 0)$

11. $P(5, -2)$ **12.** $P(22, -22)$

In Exercises 13–16, state the sign (+ or −) of **(a)** $\sin t$, **(b)** $\cos t$, and **(c)** $\tan t$ for values of t in the interval given.

13. $\left(0, \frac{\pi}{2}\right)$ **14.** $\left(\frac{\pi}{2}, \pi\right)$

15. $\left(\pi, \frac{3\pi}{2}\right)$ **16.** $\left(\frac{3\pi}{2}, 2\pi\right)$

In Exercises 17–20, determine the sign (+ or −) of the given value without the use of a calculator.

17. $\cos 143°$ **18.** $\tan 192°$

19. $\cos \frac{7\pi}{8}$ **20.** $\tan \frac{4\pi}{5}$

In Exercises 21–24, choose the point on the terminal side of θ.

21. $\theta = 45°$
(a) $(2, 2)$ **(b)** $(1, \sqrt{3})$ **(c)** $(\sqrt{3}, 1)$

22. $\theta = \frac{2\pi}{3}$
(a) $(-1, 1)$ **(b)** $(-1, \sqrt{3})$ **(c)** $(-\sqrt{3}, 1)$

23. $\theta = \frac{7\pi}{6}$
(a) $(-\sqrt{3}, -1)$ **(b)** $(-1, \sqrt{3})$ **(c)** $(-\sqrt{3}, 1)$

24. $\theta = -60°$
(a) $(-1, -1)$ **(b)** $(1, -\sqrt{3})$ **(c)** $(-\sqrt{3}, 1)$

In Exercises 25–36, evaluate without using a calculator by using ratios in a reference triangle.

25. $\cos 120°$ **26.** $\tan 300°$

27. $\sec \frac{\pi}{3}$ **28.** $\csc \frac{3\pi}{4}$

29. $\sin \frac{13\pi}{6}$ **30.** $\cos \frac{7\pi}{3}$

31. $\tan -\frac{15\pi}{4}$ **32.** $\cot \frac{13\pi}{4}$

33. $\cos \frac{23\pi}{6}$ **34.** $\cos \frac{17\pi}{4}$

35. $\sin \frac{11\pi}{3}$ **36.** $\cot \frac{19\pi}{6}$

In Exercises 37–42, find **(a)** $\sin \theta$, **(b)** $\cos \theta$, and **(c)** $\tan \theta$ for the given quadrantal angle. If the value is undefined, write "undefined."

37. $-450°$ **38.** $-270°$

39. 7π **40.** $\frac{11\pi}{2}$

41. $-\frac{7\pi}{2}$ **42.** -4π

In Exercises 43–48, evaluate without using a calculator.

43. Find $\sin \theta$ and $\tan \theta$ if $\cos \theta = \frac{2}{3}$ and $\cot \theta > 0$.

44. Find $\cos \theta$ and $\cot \theta$ if $\sin \theta = \frac{1}{4}$ and $\tan \theta < 0$.

45. Find $\tan \theta$ and $\sec \theta$ if $\sin \theta = -\frac{2}{5}$ and $\cos \theta > 0$.

46. Find $\sin \theta$ and $\cos \theta$ if $\cot \theta = \frac{3}{7}$ and $\sec \theta < 0$.

47. Find $\sec \theta$ and $\csc \theta$ if $\cot \theta = -\frac{4}{3}$ and $\cos \theta < 0$.

48. Find $\csc \theta$ and $\cot \theta$ if $\tan \theta = -\frac{4}{3}$ and $\sin \theta > 0$.

In Exercises 49–52, evaluate by using the period of the function.

49. $\sin\left(\frac{\pi}{6} + 49{,}000\pi\right)$

50. $\tan(1{,}234{,}567\pi) - \tan(7{,}654{,}321\pi)$

51. $\cos\left(\frac{5{,}555{,}555\pi}{2}\right)$

52. $\tan\left(\frac{3\pi - 70{,}000\pi}{2}\right)$

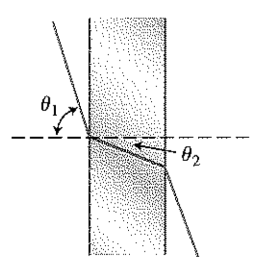
53. Group Activity Use a calculator to evaluate the expressions in Exercises 49–52. Does your calculator give the correct answers? Many calculators miss all four. Give a brief explanation of what probably goes wrong.

54. Writing to Learn Give a convincing argument that the period of $\sin t$ is 2π. That is, show that there is no smaller positive real number p such that $\sin(t + p) = \sin t$ for all real numbers t.

55. Refracted Light Light is *refracted* (bent) as it passes through glass. In the figure below θ_1 is the angle of incidence and θ_2 is the *angle of refraction.* The *index of refraction* is a constant μ that satisfies the equation

$$\sin\theta_1 = \mu\sin\theta_2.$$

If $\theta_1 = 83°$ and $\theta_2 = 36°$ for a certain piece of flint glass, find the index of refraction.



Glass

56. Refracted Light A certain piece of crown glass has an index of refraction of 1.52. If a light ray enters the glass at an angle $\theta_1 = 42°$, what is $\sin\theta_2$?

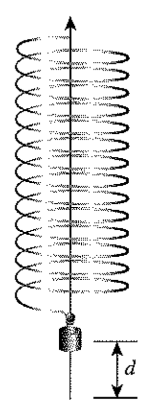
57. Damped Harmonic Motion A weight suspended from a spring is set into motion. Its dispacement d from equilibrium is modeled by the equation

$$d = 0.4e^{-0.2t}\cos 4t.$$

where d is the displacement in inches and t is the time in seconds. Find the displacement at the given time. Use radian mode.

(a) $t = 0$

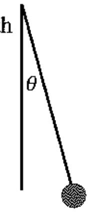
(b) $t = 3$



58. Swinging Pendulum The Columbus Museum of Science and Industry exhibits a Foucault pendulum 32 ft long that swings back and forth on a cable once in approximately 6 sec. The angle θ (in radians) between the cable and an imaginary vertical line is modeled by the equation

$$\theta = 0.25\cos t.$$

Find the measure of angle θ when $t = 0$ and $t = 2.5$.



59. Too Close for Comfort An F-15 aircraft flying at an altitude of 8000 ft passes directly over a group of vacationers hiking at 7400 ft. If θ is the angle of elevation from the hikers to the F-15, find the distance d from the group to the jet for the given angle.

(a) $\theta = 45°$ **(b)** $\theta = 90°$ **(c)** $\theta = 140°$

60. Manufacturing Swimwear Get Wet, Inc. manufactures swimwear, a seasonal product. The monthly sales x (in thousands) for Get Wet swimsuits are modeled by the equation

$$x = 72.4 + 61.7\sin\frac{\pi t}{6},$$

where $t = 1$ represents January, $t = 2$ February, and so on. Estimate the number of Get Wet swimsuits sold in January, April, June, October, and December. For which two of these months are sales the same? Explain why this might be so.

Standardized Test Questions

61. True or False If θ is an angle of a triangle such that $\cos\theta < 0$, then θ is obtuse. Justify your answer.

62. True or False If θ is an angle in standard position determined by the point $(8, -6)$, then $\sin\theta = -0.6$. Justify your answer.

You should answer these questions without using a calculator.

63. Multiple Choice If $\sin\theta = 0.4$, then $\sin(-\theta) + \csc\theta =$

(A) -0.15 **(B)** 0 **(C)** 0.15 **(D)** 0.65 **(E)** 2.1

64. Multiple Choice If $\cos\theta = 0.4$, then $\cos(\theta + \pi) =$

(A) -0.6 **(B)** -0.4 **(C)** 0.4 **(D)** 0.6 **(E)** 3.54

65. Multiple Choice The range of the function $f(t) = (\sin t)^2 + (\cos t)^2$ is

(A) $\{1\}$ **(B)** $[-1, 1]$ **(C)** $[0, 1]$

(D) $[0, 2]$ **(E)** $[0, \infty)$

66. Multiple Choice If $\cos\theta = -\frac{5}{13}$ and $\tan\theta > 0$, then $\sin\theta =$

(A) $-\frac{12}{13}$ **(B)** $-\frac{5}{12}$ **(C)** $\frac{5}{13}$ **(D)** $\frac{5}{12}$ **(E)** $\frac{12}{13}$

Explorations

In Exercises 67–70, find the value of the unique real number θ between 0 and 2π that satisfies the two given conditions.

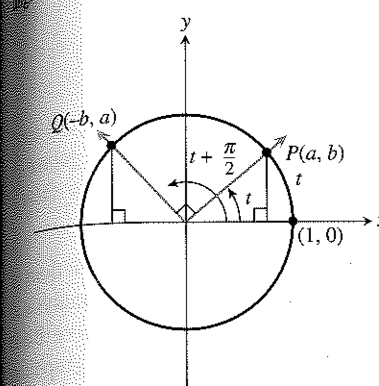
67. $\sin\theta = \frac{1}{2}$ and $\tan\theta < 0$.

68. $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta < 0$.

69. $\tan\theta = -1$ and $\sin\theta < 0$.

70. $\sin\theta = -\frac{\sqrt{2}}{2}$ and $\tan\theta > 0$.

Exercises 71–74 refer to the unit circle in this figure. Point P is on the terminal side of an angle t and point Q is on the terminal side of an angle $t + \pi/2$.



71. Using Geometry in Trigonometry Drop perpendiculars from points P and Q to the x-axis to form two right triangles. Explain how the right triangles are related.

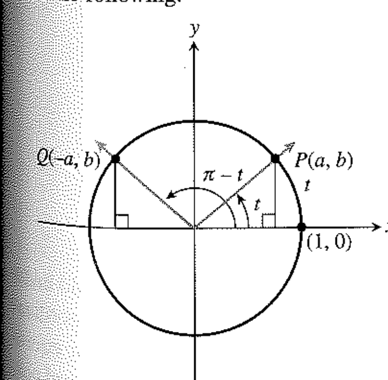
72. Using Geometry in Trigonometry If the coordinates of point P are (a, b), explain why the coordinates of point Q are $(-b, a)$.

73. Explain why $\sin\left(t + \frac{\pi}{2}\right) = \cos t$.

74. Explain why $\cos\left(t + \frac{\pi}{2}\right) = -\sin t$.

75. Writing to Learn In the figure for Exercises 71–74, t is an angle with radian measure $0 < t < \pi/2$. Draw a similar figure for an angle with radian measure $\pi/2 < t < \pi$ and use it to explain why $\sin(t + \pi/2) = \cos t$.

76. Writing to Learn Use the accompanying figure to explain each of the following.



(a) $\sin(\pi - t) = \sin t$ **(b)** $\cos(\pi - t) = -\cos t$

Extending the Ideas

77. Approximation and Error Analysis Use your grapher to complete the table to show that $\sin\theta \approx \theta$ (in radians) when $|\theta|$ is small. Physicists often use the approximation $\sin\theta \approx \theta$ for small values of θ. For what values of θ is the *magnitude of the error* in approximating $\sin\theta$ by θ less than 1% of $\sin\theta$? That is, solve the relation

$$|\sin\theta - \theta| < 0.01\,|\sin\theta|.$$

(*Hint:* Extend the table to include a column for values of $\frac{|\sin\theta - \theta|}{|\sin\theta|}$.)

θ	$\sin\theta$	$\sin\theta - \theta$
−0.03		
−0.02		
−0.01		
0		
0.01		
0.02		
0.03		

78. Proving a Theorem If t is any real number, prove that $1 + (\tan t)^2 = (\sec t)^2$.

Taylor Polynomials Radian measure allows the trigonometric functions to be approximated by simple polynomial functions. For example, in Exercises 79 and 80, sine and cosine are approximated by Taylor polynomials, named after the English mathematician Brook Taylor (1685–1731). Complete each table showing a Taylor polynomial in the third column. Describe the patterns in the table.

79.

θ	$\sin\theta$	$\theta - \frac{\theta^3}{6}$	$\sin\theta - \left(\theta - \frac{\theta^3}{6}\right)$
−0.3	−0.295...		
−0.2	−0.198...		
−0.1	−0.099...		
0	0		
0.1	0.099...		
0.2	0.198...		
0.3	0.295...		

80.

θ	$\cos\theta$	$1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$	$\cos\theta - \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right)$
−0.3	0.955...		
−0.2	0.980...		
−0.1	0.995...		
0	1		
0.1	0.995...		
0.2	0.980...		
0.3	0.955...		